Notes on the Nested Logit Demand Model

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These notes derive formulas for the market shares, first derivatives, and second derivatives of the nested logit model. Section 1 examines on a one-level nesting structure, and Section 2 examines a two-level nesting structure. The notes were originally prepared as part of the Miller and Weinberg (2017) research project,[1] though they have proven useful in a number of other projects as well. We have prepared corresponding functions in R that return the market shares and derivatives given the structural parameters as inputs.

1 Single Level Nested Logit

1.1 Framework and notation

Here we use the Berry (1994) notation, which itself derives from the Cardell (1991) formulation of the nested logit model. Let the products be grouped into $G + 1$ exhaustive and mutually exclusive sets, $g = 0, 1, \ldots G$. Denote the set of products in group g as $\mathscr{J}_g$. The outside good, $j = 0$, is the only member of group 0. For product $j \in \mathscr{J}_g$ let the utility of consumer i be

$$u_{ij} = \delta_j + \varsigma_{ig} + (1 - \sigma)\epsilon_{ij}, \tag{1}$$

where $\delta_j = x_j'\beta + \alpha p_j + \xi_j$ and ϵ_{ij} is iid extreme value. The idiosyncratic group preference, ς_{ig}, follows the unique distribution such that $\varsigma_{ig} + (1 - \sigma)\epsilon_{ij}$ is also an extreme value random variable, and the parameter σ, with $0 \leq \sigma < 1$, characterizes the correlation of utilities that a consumer experiences among the products in the same group. With $\sigma = 0$, the model collapses to a standard logit.

If product j is in group g, i.e., $j \in \mathscr{J}_g$, then the selection probability of product j conditional on group g being selected equals

$$\overline{s}_{j|g} = \frac{\exp\left(\frac{\delta_j}{1-\sigma}\right)}{D_g}, \tag{2}$$

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[1]Nathan Miller and Matthew Weinberg (2017), "Understanding the Price Effects of the MillerCoors Joint Venture." *Econometrica*, Vol. 85, No. 6, 1763-1791.

where

$$D_g = \sum_{k \in \mathscr{J}_g} \exp\left(\frac{\delta_k}{1-\sigma}\right). \tag{3}$$

The unconditional selection probability of the group g being selected equals

$$\overline{s}_g = \frac{D_g^{(1-\sigma)}}{\sum_g D_g^{(1-\sigma)}}. \tag{4}$$

Thus the unconditional selection probability of product j, in group g, equals

$$s_j = \overline{s}_{j|g}\overline{s}_g = \frac{\exp\left(\frac{\delta_j}{1-\sigma}\right)}{D_g^{\sigma}\left[\sum_g D_g^{(1-\sigma)}\right]}. \tag{5}$$

1.2 First derivatives

The price derivatives enter estimation through the supply-side moments. From a programming standpoint it makes sense to have a function that calculates derivatives with respect to the mean valuation, i.e. $\partial s_j/\partial\delta_k$, which given the linearity of δ can be transformed easily to recover the derivatives with respect to any price or non-price demand shifter.

Group share derivatives. These can be represented by a $G \times J$ matrix. The derivative of a group share with respect to the mean valuation of a product *inside* that group is given by

$$\frac{\partial\overline{s}_g}{\partial\delta_k} = s_k(1-\overline{s}_g). \tag{6}$$

The derivative of a group share with respect to the mean valuation of a product *outside* that group is given by

$$\frac{\partial\overline{s}_g}{\partial\delta_k} = -\overline{s}_g s_k. \tag{7}$$

Conditional share derivatives. These can be represented by a $J \times J$ matrix. The derivative of j's conditional share with respect to its own mean valuation is given by

$$\frac{\partial\overline{s}_{j|g}}{\partial\delta_j} = \frac{1}{1-\sigma}\overline{s}_{j|g}(1-\overline{s}_{j|g}) \tag{8}$$

The derivative of j's conditional share with respect to product k's mean valuation, for j and k both in group g, is given by

$$\frac{\partial\overline{s}_{j|g}}{\partial\delta_k} = -\frac{1}{1-\sigma}\overline{s}_{j|g}\overline{s}_{k|g} \tag{9}$$

The derivative of j's conditional share with respect to product k's mean valuation, for j and k in different groups, is given by

$$\frac{\partial\overline{s}_{j|g}}{\partial\delta_k} = 0 \tag{10}$$

Unconditional share derivatives. These can be represented by a $J \times J$ matrix, where the (j,k) element of the matrix is given by

$$\frac{\partial s_j}{\partial \delta_k} = \frac{\partial \overline{s}_{j|g}}{\partial \delta_k}\overline{s}_g + \overline{s}_{j|g}\frac{\partial \overline{s}_g}{\partial \delta_k}. \tag{11}$$

This matrix can be easily constructed, without looping, using the matrices of shares and group and conditional derivatives defined above. However, these derivatives can also be written as a function of only shares, shown below.

The derivative of product j's share, for $j \in \mathscr{J}_g$, with respect to its own mean valuation is given by

$$\frac{\partial s_j}{\partial \delta_j} = \frac{1}{1-\sigma} s_j(1 - \sigma\overline{s}_{j|g} - (1-\sigma)s_j). \tag{12}$$

The derivative of product j's share with respect to product k's mean valuation, for j and k both in group g, equals

$$\frac{\partial s_j}{\partial \delta_k} = -s_k\left(s_j + \frac{\sigma}{1-\sigma}\overline{s}_{j|g}\right) \tag{13}$$

The above formulation is correct but more difficult to use in obtaining cross-cross second derivatives because the symmetry between j and k isn't transparent. More helpful for those is the equivalent expression

$$\frac{\partial s_j}{\partial \delta_k} = -s_k s_j\left(1 + \frac{\sigma}{1-\sigma}\frac{1}{\overline{s}_g}\right). \tag{14}$$

The derivative of product j's share with respect to product k's mean valuation, for j and k in different groups, equals

$$\frac{\partial s_j}{\partial \delta_k} = -s_j s_k \tag{15}$$

1.3 Second derivatives

The second derivatives of unconditional shares can be expressed as a $J \times J \times J$ array. The (j,k,l) element of this array takes the form

$$\frac{\partial^2 s_j}{\partial \delta_k \partial \delta_l} = \frac{\partial^2 \overline{s}_{j|g}}{\partial \delta_k \partial \delta_l}\overline{s}_g + \frac{\partial \overline{s}_{j|g}}{\partial \delta_l}\frac{\partial \overline{s}_g}{\partial \delta_k} \tag{16}$$

$$+ \frac{\partial \overline{s}_{j|g}}{\partial \delta_k}\frac{\partial \overline{s}_g}{\partial \delta_l} + \overline{s}_{j|g}\frac{\partial^2 \overline{s}_g}{\partial \delta_k \partial \delta_l} \tag{17}$$

This can be calculated using previously defined arrays and the two conditional second derivatives defined next.

Product conditional share second derivatives These can be summarized as a $J \times J \times J$ array where, if $j = k$, element (j,k,l) is

$$\frac{\partial^2 \overline{s}_{j|g}}{\partial \delta_k \partial \delta_l} = \frac{1}{1-\sigma}\frac{\partial \overline{s}_{j|g}}{\partial \delta_l}(1 - 2\overline{s}_{j|g}) \tag{18}$$

and if $j \neq k$, then element (j, k, l) is

$$\frac{\partial^2 \overline{s}_{j|g}}{\partial \delta_k \partial \delta_l} = -\frac{1}{1-\sigma}\left(\frac{\partial \overline{s}_{j|g}}{\partial \delta_l}\overline{s}_{k|g} + \overline{s}_{j|g}\frac{\partial \overline{s}_{k|g}}{\partial \delta_l}\right) \tag{19}$$

If either k or l are not in the same group as j, then these derivatives will equal zero.

Group share second derivatives These can be summarized as a $G \times J \times J$ array where, if k is in group g, element (g, k, l) is

$$\frac{\partial^2 \overline{s}_g}{\partial \delta_k \partial \delta_l} = \frac{\partial s_k}{\partial \delta_l}(1 - \overline{s}_g) - s_k \frac{\partial \overline{s}_g}{\partial \delta_l} \tag{20}$$

and if k is not in group g, then element (g, k, l) is

$$\frac{\partial^2 \overline{s}_g}{\partial \delta_k \partial \delta_l} = -\left(\frac{\partial \overline{s}_g}{\partial \delta_l} s_k + \overline{s}_g \frac{\partial s_k}{\partial \delta_l}\right) \tag{21}$$

While the expressions above are sufficient for programming purposes, it is also possible to further simplify the expressions for the unconditional share second derivatives.

1. **Own-own derivative.** The second derivative of j's share with respect to its valuation is given by

$$\begin{aligned}
\frac{\partial^2 s_j}{\partial^2 \delta_j} &= \left(\frac{1}{1-\sigma} - \frac{\sigma}{1-\sigma}\overline{s}_{j|g} - 2s_j\right)\frac{\partial s_j}{\partial \delta_j} - \frac{\sigma}{1-\sigma} s_j \frac{\partial \overline{s}_{j|g}}{\partial \delta_j} && (22)\\
&= \left(\frac{1}{1-\sigma} - \frac{\sigma}{1-\sigma}\overline{s}_{j|g} - 2s_j\right)\left(\frac{1}{1-\sigma} - \frac{\sigma}{1-\sigma}\overline{s}_{j|g} - s_j\right) s_j \\
&\quad - \frac{\sigma}{(1-\sigma)^2} s_j \overline{s}_{j|g}(1 - \overline{s}_{j|g}) && (23)
\end{aligned}$$

2. **Own-cross derivative.** The second derivative of j's share with respect to the valuation of its product and that of another product k, such j and k are in different groups, is given by

$$\begin{aligned}
\frac{\partial^2 s_j}{\partial \delta_j \partial \delta_k} &= -\left(\frac{\partial s_j}{\partial \delta_j} s_k + \frac{\partial s_k}{\partial \delta_j} s_j\right) && (24)\\
&= -\frac{1}{1-\sigma} s_j s_k (1 - \sigma \overline{s}_{j|g} - (1-\sigma)s_j) + s_j^2 s_k && (25)
\end{aligned}$$

3. **Own-cross derivative.** The second derivative of j's share with respect to the valuation of its product and that of another product k, such j and k are in the same group, is given by

$$\begin{aligned}
\frac{\partial^2 s_j}{\partial \delta_j \partial \delta_k} &= -\frac{\partial s_k}{\partial \delta_j}\left(s_j + \frac{\sigma}{1-\sigma}\overline{s}_{j|g}\right) - s_k \left(\frac{\partial s_j}{\partial \delta_j} + \frac{\sigma}{1-\sigma}\frac{\partial \overline{s}_{j|g}}{\partial \delta_j}\right) && (26)\\
&= s_j \left(s_k + \frac{\sigma}{1-\sigma}\overline{s}_{k|g}\right)\left(s_j + \frac{\sigma}{1-\sigma}\overline{s}_{j|g}\right) \\
&\quad - s_k \left(\frac{1}{1-\sigma} s_j (1 - \sigma \overline{s}_{j|g} - (1-\sigma)s_j) + \frac{\sigma}{(1-\sigma)^2}\overline{s}_{j|g}(1 - \overline{s}_{j|g})\right) && (27)
\end{aligned}$$

4. **Cross-cross derivative.** The second derivative of j's share with respect to the valuation of two products, such that all three products involved are in different nests, is given by

$$\frac{\partial^2 s_j}{\partial\delta_k\partial\delta_l} = 2s_js_ks_l \tag{28}$$

5. **Cross-cross derivative.** The second derivative of j's share with respect to the valuation of two products k and l, such k and l are in a group g that is different than j's group, is given by

$$\frac{\partial^2 s_j}{\partial\delta_k\partial\delta_l} = s_js_ks_l + s_js_ks_l\left(1+\frac{\sigma}{1-\sigma}\frac{1}{\overline{s}_g}\right) \tag{29}$$

6. **Cross-cross derivative.** The second derivative of j's share with respect to the valuation of two products k and l, such all three products are in the same group g, is given by

$$\begin{aligned}\frac{\partial^2 s_j}{\partial\delta_k\partial\delta_l} &= -\frac{\partial s_k}{\partial\delta_l}s_j\left(1+\frac{\sigma}{1-\sigma}\frac{1}{\overline{s}_g}\right) - s_k\frac{\partial s_j}{\partial\delta_l}\left(1+\frac{\sigma}{1-\sigma}\frac{1}{\overline{s}_g}\right) \\ &\quad +s_ks_j\left(\frac{\sigma}{1-\sigma}\right)\frac{\partial\overline{s}_g}{\partial\delta_l}/\overline{s}_g^2 \qquad (30)\\ &= 2s_js_ks_l\left(1+\frac{\sigma}{1-\sigma}\frac{1}{\overline{s}_g}\right)^2 + s_js_ks_l\frac{\sigma}{1-\sigma}\frac{1-\overline{s}_g}{\overline{s}_g^2} \qquad (31)\end{aligned}$$

7. **Cross-cross derivative.** The second derivative of j's share with respect to the valuation of two products k and l, such all j and k are in a group g that is different than l's group, is given by

$$\frac{\partial^2 s_j}{\partial\delta_k\partial\delta_l} = s_js_ks_l\left(2+\frac{\sigma}{1-\sigma}\frac{1}{\overline{s}_g}\right) \tag{32}$$

8. **Cross-cross derivative.** The second derivative of j's share with respect to the valuation of products k, such j and k are in a group g, is given by

$$\begin{aligned}\frac{\partial^2 s_j}{\partial^2\delta_k} &= -\frac{\partial s_k}{\partial\delta_k}\left(s_j+\frac{\sigma}{1-\sigma}\overline{s}_{j|g}\right) - s_k\left(\frac{\partial s_j}{\partial\delta_k}-\frac{\sigma}{1-\sigma}\frac{\partial\overline{s}_{j|g}}{\partial\delta_k}\right) \qquad (33)\\ &= -\frac{1}{1-\sigma}s_k(1-\sigma\overline{s}_{k|g}-(1-\sigma)s_k)\left(s_j+\frac{\sigma}{1-\sigma}\overline{s}_{j|g}\right) \\ &\quad +s_k^2\left(s_j+\frac{\sigma}{1-\sigma}\overline{s}_{j|g}\right)+\frac{\sigma}{1-\sigma}\frac{1}{1-\sigma}s_k\overline{s}_{j|g}\overline{s}_{k|g} \qquad (34)\\ & \qquad (35)\end{aligned}$$

9. **Cross-cross derivative.** The second derivative of j's share with respect to the valuation of products k, such k is in group g but j is not, is given by

$$\frac{\partial^2 s_j}{\partial^2\delta_k} = s_js_k\left(s_k-\frac{1}{1-\sigma}\left(1-\sigma\overline{s}_{k|g}-(1-\sigma)s_k\right)\right) \tag{36}$$

2 Two Level Nested Logit

Let the products be partitioned into $G+1$ groups, $g = 0, \ldots, G$. Further let each group g be partitioned into H_g subgroups, $h = 1, \ldots H_g$. Denote the set of subgroups in group g as $\mathscr{H}_g$ and the set of products in subgroup h of group g as $\mathscr{J}_{h,g}$. The outside good, $j = 0$, is the only member of group 0. For product $j \in \mathscr{J}_{h,g}$ let the utility of consumer i be

$$u_{ij} = \delta_j + \varsigma_i^g + (1-\sigma_2)\varsigma_i^h + (1-\sigma_1)(1-\sigma_2)\epsilon_{ij}, \tag{37}$$

where $\delta_j = x_j'\beta + \alpha p_j + \xi_j$ and ϵ_{ij} is iid extreme value. In this context, ς_i^g and ς_i^h each follow the unique distributions such that $[\varsigma_i^h + (1-\sigma_1)(1-\sigma_2)\epsilon_{ij}]$ and $[\varsigma_i^g + (1-\sigma_2)\varsigma_i^h + (1-\sigma_1)(1-\sigma_2)\epsilon_{ij}]$ are both extreme value random variables. McFadden (1978) proves that this is consistent with random utility maximization if $1 \geq \sigma_1 \geq \sigma_2 \geq 0$. The nesting parameter σ_1 captures the correlation of utilities that consumers experience among products in the same subgroup and the parameter σ_2 captures the same concept for the groups. The model collapses to a one-level nested logit with groups as nests if $\sigma_1 = \sigma_2$. The model collapses to a one-level nested logit with sub-groups as nests if $\sigma_2 = 0$.

The Berry (1994) inversion provides that

$$\delta_j = \log(s_j) - \log(s_0) - \sigma_1 \log(\overline{s}_{j|hg}) - \sigma_2 \log(\overline{s}_{h|g}), \tag{38}$$

where s_j is the unconditional selection probability of product j, $\overline{s}_{j|hg}$ is the selection probability of product j condition on its subgroup (and group) being selected, and $\overline{s}_{h|g}$ is the probability that product j's subgroup is selected conditional on its group being selected.

In deriving expressions for these selection probabilities the following equations are useful:

$$I_{hg} = (1-\sigma_1) \log \sum_{k \in \mathscr{J}_{h,g}} \exp\left(\frac{\delta_k}{1-\sigma_1}\right) \tag{39}$$

$$I_g = (1-\sigma_2) \log \sum_{h \in \mathscr{H}_g} \exp\left(\frac{I_{hg}}{1-\sigma_2}\right) \tag{40}$$

$$I = \log(1 + \sum_g \exp(I_g)). \tag{41}$$

The last equation places the outside option among the groups – it is the source of the "1". With these equations in hand, the selection probability of product j conditional on its subgroup being selected is given by

$$\overline{s}_{j|hg} = \frac{\exp\left(\frac{\delta_j}{1-\sigma_1}\right)}{\exp\left(\frac{I_{hg}}{1-\sigma_1}\right)} \tag{42}$$

The probability that subgroup h is selected conditional on its group being selected is given by

$$\overline{s}_{h|g} = \frac{\exp\left(\frac{I_{hg}}{1-\sigma_2}\right)}{\exp\left(\frac{I_g}{1-\sigma_2}\right)}. \tag{43}$$

The probability that group g is selected equals

$$\overline{s}_g = \frac{\exp(I_g)}{\exp(I)}. \tag{44}$$

Lastly the unconditional selection probability of product j takes the form

$$s_j = \overline{s}_{j|hg}\overline{s}_{h|g}\overline{s}_g. \tag{45}$$

2.1 First derivatives of Conditional Shares

The derivatives of the conditional shares with respect to delta are useful in obtaining, in a programming-friendly way, the first and second derivatives of the unconditional shares with respect to delta.

Product conditional share derivatives. These derivatives can be captured with a $J \times J$ matrix. The derivative of j's share, conditional on its subgroup being selected, with respect to its mean valuation is given by

$$\frac{\partial \overline{s}_{j|hg}}{\partial \delta_j} = \frac{1}{1-\sigma_1}\overline{s}_{j|hg}(1-\overline{s}_{j|hg}) \tag{46}$$

The derivative of j's conditional share with respect to the valuation of a product in the same subgroup is given by

$$\frac{\partial \overline{s}_{j|hg}}{\partial \delta_k} = -\frac{1}{1-\sigma_1}\overline{s}_{j|hg}\overline{s}_{k|hg} \tag{47}$$

The derivative of j's conditional share with respect to the valuation of a product in a different group or subgroup is given by

$$\frac{\partial \overline{s}_{j|hg}}{\partial \delta_k} = 0 \tag{48}$$

Subgroup conditional share derivatives. These derivatives can be captured with a $H \times J$ matrix. The derivative of h's share, conditional on its group being selected, with respect to the mean valuation of a product in h is given by

$$\frac{\partial \overline{s}_{h|g}}{\partial \delta_k} = \frac{1}{1-\sigma_2}\overline{s}_{k|hg}\overline{s}_{h|g}(1-\overline{s}_{h|g}) \tag{49}$$

The derivative of h's conditional share with respect to the valuation of a product in a different subgroup $h^* \neq h$ but the same group is given by

$$\frac{\partial \overline{s}_{h|g}}{\partial \delta_k} = -\frac{1}{1-\sigma_2}\overline{s}_{h|g}\overline{s}_{k|h^*g}\overline{s}_{h^*|g} \tag{50}$$

The derivative of h's conditional share with respect to the valuation of a product in a different group is given by

$$\frac{\partial \overline{s}_{h|g}}{\partial \delta_k} = 0 \tag{51}$$

Group share derivatives. These derivatives can be captured in a $G \times J$ matrix. The derivative the group share with respect to the mean valuation of a product within the group is given by:

$$\frac{\partial \overline{s}_g}{\partial \delta_k} = s_k(1 - \overline{s}_g). \tag{52}$$

The derivative of the group share with respect to the mean valuation of a product *outside* the group is given by

$$\frac{\partial \overline{s}_g}{\partial \delta_k} = -\overline{s}_g s_k. \tag{53}$$

2.2 Unconditional selection probability first derivatives

These can be represented by a $J \times J$ matrix. To avoid programming loops, the matrix can be constructed using the expressions above. The (j,k) element of the matrix is given by

$$\frac{\partial s_j}{\partial \delta_k} = \frac{\partial \overline{s}_{j|hg}}{\partial \delta_k}\overline{s}_{h|g}\overline{s}_g + \overline{s}_{j|hg}\frac{\partial \overline{s}_{h|g}}{\partial \delta_k}\overline{s}_g + \overline{s}_{j|hg}\overline{s}_{h|g}\frac{\partial \overline{s}_g}{\partial \delta_k} \tag{54}$$

This can be taken further, plugging in expressions for the derivatives and simplifying. For instance, that's the procedure outlined in the first section above and what we did in initial programming. But that requires nested looping in the code which would have been prohibitively taxing on estimation time.

2.3 Unconditional second derivatives

The unconditional second derivatives can be represented in a $J \times J \times J$ array. The (j^{th}, k^{th}, l^{th}) element of this array takes the form

$$\begin{aligned}
\frac{\partial^2 s_j}{\partial \delta_k \partial \delta_l} &= \frac{\partial^2 \overline{s}_{j|hg}}{\partial \delta_k \partial \delta_l}\overline{s}_{h|g}\overline{s}_g + \frac{\partial \overline{s}_{j|hg}}{\partial \delta_l}\frac{\partial \overline{s}_{h|g}}{\partial \delta_k}\overline{s}_g + \frac{\partial \overline{s}_{j|hg}}{\partial \delta_l}\overline{s}_{h|g}\frac{\partial \overline{s}_g}{\partial \delta_k} && (55)\\
&+ \frac{\partial \overline{s}_{j|hg}}{\partial \delta_k}\frac{\partial \overline{s}_{h|g}}{\partial \delta_l}\overline{s}_g + \overline{s}_{j|hg}\frac{\partial^2 \overline{s}_{h|g}}{\partial \delta_k \partial \delta_l}\overline{s}_g + \overline{s}_{j|hg}\frac{\partial \overline{s}_{h|g}}{\partial \delta_l}\frac{\partial \overline{s}_g}{\partial \delta_k} && (56)\\
&+ \frac{\partial \overline{s}_{j|hg}}{\partial \delta_k}\overline{s}_{h|g}\frac{\partial \overline{s}_g}{\partial \delta_l} + \overline{s}_{j|hg}\frac{\partial \overline{s}_{h|g}}{\partial \delta_k}\frac{\partial \overline{s}_g}{\partial \delta_l} + \overline{s}_{j|hg}\overline{s}_{h|g}\frac{\partial^2 \overline{s}_g}{\partial \delta_k \partial \delta_l} && (57)
\end{aligned}$$

This can be constructed, with no looping, from arrays already defined plus some conditional second derivatives that will be derived next. The key thing for the programming is to get each of the input arrays properly transposed.

2.4 Conditional second derivatives

Product conditional share derivatives. These can be summarized in a $J \times J \times J$ array. If $j = k$ then the (j,k,l) element equals

$$\frac{\partial^2 \overline{s}_{j|hg}}{\partial \delta_k \partial \delta_l} = \frac{1}{1-\sigma_1}\left(\frac{\partial \overline{s}_{j|hg}}{\partial \delta_l} - 2\frac{\partial \overline{s}_{j|hg}}{\partial \delta_l}\overline{s}_{j|hg}\right) \tag{58}$$

If $j \neq k$ then the (j, k, l) element equals

$$\frac{\partial^2 \overline{s}_{j|hg}}{\partial \delta_k \partial \delta_l} = -\frac{1}{1-\sigma_1}\left(\frac{\partial \overline{s}_{j|hg}}{\partial \delta_l}\overline{s}_{k|hg} + \overline{s}_{j|hg}\frac{\partial \overline{s}_{k|hg}}{\partial \delta_l}\right) \tag{59}$$

If k or l are not in h these go to zero immediately with no more effort because that is incorporated into the first derivative matrices.

Subgroup conditional share derivatives. These can be summarized in a $H \times J \times J$ array. For incorporation into the unconditional second derivatives, the first dimension needs to be expanded appropriately such that each h row is allocated to each of the products that are in the h subgroup. Let k be in subgroup h. Then

$$\frac{\partial^2 \overline{s}_{h|g}}{\partial \delta_k \partial \delta_l} = \frac{1}{1-\sigma_2}\left(\frac{\partial \overline{s}_{k|hg}}{\partial \delta_l}\overline{s}_{h|g}\left(1-\overline{s}_{h|g}\right) + \overline{s}_{k|hg}\frac{\partial \overline{s}_{h|g}}{\partial \delta_l}\left(1-2\overline{s}_{h|g}\right)\right) \tag{60}$$

If instead product k is in subroup $h^* \neq h$ but is in group g then

$$\frac{\partial^2 \overline{s}_{h|g}}{\partial \delta_k \partial \delta_l} = -\frac{1}{1-\sigma_2}\left(\frac{\partial \overline{s}_{k|h^*g}}{\partial \delta_l}\overline{s}_{h|g}\overline{s}_{h^*|g} + \overline{s}_{k|h^*g}\frac{\partial \overline{s}_{h|g}}{\partial \delta_l}\overline{s}_{h^*|g} + \overline{s}_{k|h^*g}\overline{s}_{h|g}\frac{\partial \overline{s}_{h^*|g}}{\partial \delta_l}\right) \tag{61}$$

(62)

If k or l are not in g then

$$\frac{\partial^2 \overline{s}_{h|g}}{\partial \delta_k \partial \delta_l} = 0 \tag{63}$$

Group share derivatives. These can be summarized in a $G \times J \times J$ array. For incorporation into the unconditional second derivatives the first dimension needs to be expanded appropriately such that each g row is allocated to each of the products that are in the g group. Let k be in group g. Then

$$\frac{\partial^2 \overline{s}_{g}}{\partial \delta_k \partial \delta_l} = \frac{\partial s_k}{\partial \delta_l}(1-\overline{s}_g) - s_k\frac{\partial \overline{s}_g}{\partial \delta_l} \tag{64}$$

If instead k is not in group g then

$$\frac{\partial^2 \overline{s}_{g}}{\partial \delta_k \partial \delta_l} = -\left(\frac{\partial s_k}{\partial \delta_l}\overline{s}_g + s_k\frac{\partial \overline{s}_g}{\partial \delta_l}\right) \tag{65}$$