Mergers, Entry, and Consumer Welfare*

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FIRST DRAFT—COMMENTS WELCOME

Abstract

We analyze mergers and entry in a differentiated products oligopoly model of price competition. We prove that any merger among incumbents is unprofitable if it spurs entry sufficient in magnitude to preserve consumer surplus. Thus, mergers occur in equilibrium only if barriers limit entry. Numerical simulations indicate that with profit-neutral mergers—the best-case for consumers—entry mitigates under 30 percent of the adverse price effects and, in most cases, under 50 percent of the consumer surplus loss. The results suggest a limited and conditional role for entry analysis in merger review.

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*The analysis and conclusions set forth are those of the authors and do not indicate concurrence by other members of the Board research staff or the Board of Governors.

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1 Introduction

The antitrust review of mergers in the Unites States follows a set of evaluation criteria summarized in the 2010 Horizontal Merger Guidelines of the Department of Justice and Federal Trade Commission. An important step examines entry—specifically, the question of whether entry is easy enough that the merging firms would find it unprofitable to raise prices or otherwise reduce competition. In this framework, the existence of entry that meets the standard of being timely, likely, and sufficient in its magnitude to alleviate consumer harm is grounds for approving a merger.[1]

In this paper, we reexamine mergers and entry in a differentiated products oligopoly model of price competition. We obtain a simple result: post-merger entry is never sufficient to fully mitigate the price increases and consumer surplus loss caused by profitable mergers. The result arises due to a selection effect, namely that entry sufficient to offset adverse competitive effects also renders mergers unprofitable. As firms would typically merge only if it is profitable, the existence of a merger proposal suggests a perception among the merging firms that barriers obstruct entry.[2]

Our analysis is based on a three-stage game of oligopoly in which (1) two market incumbents decide whether to merge, (2) a market outsider decides whether to incur a cost to enter the market, and (3) prices are set according to differentiated products Bertrand competition with logit demand.[3] If the merger decision does not affect the entry decision—which occurs if entry costs are low enough or high enough—then merger is both profitable and harmful for consumer welfare. More interesting is the case of *merger-induced entry*, in which entry is profitable if and only if a merger occurs. Thus, our focus is on whether mergers can coexist with merger-induced entry that leaves consumers unharmed relative to a scenario without merger or entry.[4]

To make the analysis tractable, we formulate the model as an aggregative game, following Nocke and Schutz (2018a). This provides a critical simplification. In par-

[1]See Shapiro (2010) for a useful discussion of the 2010 Horizontal Merger Guidelines. The standard of timely, likely, and sufficient also appears in the 2004 Merger Guidelines of the European Union.

[2]Farrell and Shapiro (1990) develop a similar revealed-preference argument in the context of merger efficiencies with Cournot competition: efficiencies may be possible to infer from the proposal of a merger that otherwise appears unprofitable. Werden (1991) provides three reasons unprofitable mergers might sometimes be observed.

[3]The Bertrand logit has long been a workhorse model for the antitrust analysis of horizontal mergers in differentiated products settings (Werden and Froeb (1994, 2002); Miller and Sheu (2020)). It also provides the foundation for seminal academic research based on the more flexible random coefficients logit model (e.g. Berry et al. (1995); Nevo (2001)).

[4]The 2010 Horizontal Merger Guidelines, §9, states: "[t]his section concerns entry or adjustments to pre-existing entry plans that are induced by the merger."

ticular, the contribution of each firm to equilibrium outcomes is fully determined by its "type," a scalar which captures the quality and marginal cost of its products. With merger-induced entry, the profitability of merger decreases with the type of the entrant. Thus, there exists some critical entrant type that makes the merger profit-neutral.

We refer to this critical entrant type as characterizing *best-case entry* because it provides the greatest possible consumer surplus in the event of merger with entry. This raises a natural question: can merger followed by best-case entry preserve (or improve upon) pre-merger levels of consumer surplus? Analyzing the structure of the model provides an answer in the negative: merger-induced entry is never sufficient to eliminate the adverse consumer surplus effects of profitable mergers. Restated, there is no subgame perfect equilibrium (SPE) in which a merger occurs and consumer surplus does not fall as a result. The result extends to incumbent repositioning, entry by any number of outsiders, or any combination of repositioning and entry—none of these eliminate the adverse effects of profitable mergers.

We use numerical analyses to inform the mitigation of consumer harm that occurs due to merger with best-case entry. First, we develop a partial calibration routine with which merger effects can be obtained from market shares.[5] We then consider a range of hypothetical mergers, varying the size of the merging firms, and obtain the type of the best-case entrant in each case. The results suggest that best-case entry never counteracts more than 30 percent of the merging firms' price increase. If there are few incumbents and the merging firms are small, then best-case entry can mitigate nearly 80 percent of consumer surplus loss, mainly by increasing product diversity. Otherwise, the mitigation of consumer surplus loss tends to be well less than 50 percent. Finally, mergers with best-case entry can increase total welfare relative to the pre-merger equilibrium if the merging firms are sufficiently small.

Our results suggest a more nuanced and conditional role for entry analysis than is discussed in the 2010 Horizontal Merger Guidelines. In particular, we verify numerically that a *combination* of merger efficiencies and merger-induced entry can be compatible with profitable mergers that increase consumer surplus.[6] An implication is that entry should be evaluated in conjunction with efficiencies in merger review because entry alone is not sufficient to preserve consumer surplus. Our results also

[5]The partial calibration routine itself provides a contribution to the antitrust literature by relaxing the data requirements for merger simulation in settings for which logit demand is reasonable. We extend partial calibration to nested logit in Miller and Sheu (2020). See also Nocke and Whinston (2019).

[6]Previous research points out that merger efficiencies reduce the scope for profitable entry (Cabral (2003); Erkal and Piccinin (2010)).

call into question whether antitrust authorities should bear the burden of enumerating entry barriers, as a merger proposal may reveal a belief that barriers exist.

Our research builds on a number of articles that consider the relationship between mergers and entry. The closest is Anderson et al. (2018), which examines mergers in an oligopolistic market with a competitive fringe, also using an aggregative games framework. Under a free entry assumption, they show that mergers do not harm consumers in the long run, but also do not occur as they are unprofitable. Our research differs in that it examines profitable mergers and best-case entry. This refocusing of the analysis obtains novel policy insights. Further, we provide numerical results to help quantify the extent to which entry could mitigate adverse merger effects.

Also relevant are Werden and Froeb (1998) and Spector (2003), which develop revealed preference arguments similar to ours. The first of these addresses Bertrand competition with logit demand but does not provide analytical results. Rather, the authors conduct a Monte Carlo experiment and find that mergers with entry are profitable only if the entrant is highly inefficient. This accords with our numerical results, as we find that the best-case entrant typically has a type that is substantially less than that of the merging firms. Spector (2003) examines a Cournot model, which is a simpler case because entry does not affect product diversity.

Our paper contributes to recent research that applies the aggregative games framework of Nocke and Schutz (2018a) to questions relevant to antitrust policy. Nocke and Schutz (2018b) provides conditions under which the change in the Herfindahl Index approximates the market power effects of a merger, and also examines merger efficiencies. Garrido (2019) explores endogenous product portfolios in a dynamic game. Parameterizations based on the ready-to-eat cereal industry suggest that allowing for endogenous products amplifies the consumer harm of mergers because the merging firms' portfolio reductions are greater than the portfolio expansions of non-merging competitors. Nocke and Whinston (2019) derive the efficiencies necessary to counterbalance adverse merger effects. The formula requires only the merging firms' market shares and thus relates to our partial calibration routine.

As with this other research, we lean heavily on the reformulation of Bertrand competition with logit demand as an aggregative game. Whether our results extend to other differentiated-products demand systems, such as the almost ideal demand system (e.g., Deaton and Muellbauer (1980)) or the random coefficient logit (e.g., Berry et al. (1995)), is an open question. Because the attributes of a firm in these other models cannot be summarized with a single parameter (the type), a unique best-case entrant may not exist, greatly complicating a general analysis. Still, it may be possible to develop specific results in merger review, tailored to a particular market, using numerical techniques analogous to the ones we employ.

The remainder of the paper is organized into two parts. Section 2 outlines the model and provides the main theoretical result. Section 3 discusses partial calibration and the numerical results. It includes an example in which entry and efficiencies combine to eliminate the adverse consumer surplus effects of a profitable merger.

2 Theoretical Model

We use a logit Bertrand setup, expressed as an aggregative game following Nocke and Schutz (2018a), and embed it in a three-stage entry model. In the first stage of the game, two firms decide whether to merge. After observing the outcome of that decision, in the second stage an additional firm decides whether to enter. Then in the third stage, firms sell to consumers in Bertrand competition.

2.1 Bertrand Competition Stage

We begin by describing the demand and supply setup in the final stage of the game. The profits earned from these sales determine the payoffs for the decisions made in earlier stages. Demand takes the multinomial logit form, and firms set prices simultaneously. Firms have knowledge of the consumer demand function and observe the quality and marginal costs for their own and for rivals' products.

Let there be a finite and nonempty set of differentiated products $\mathcal{J}$ available to consumers. Each consumer purchases a single product or forgoes a purchase by selecting the outside good. Let the indirect utility that consumer i receives from product $j \in \mathcal{J}$ be given by $u_{ij} = \delta_j - \alpha p_j + \epsilon_{ij}$, where δ_j is the quality of product j, and p_j is its price. The α measures sensitivity to price, whereas ϵ_{ij} is a consumer-specific preference shock. The utility of the outside good is normalized such that $u_{i0} = \epsilon_{i0}$.

We assume that the preference shocks are independently and identically distributed with a Type 1 extreme value distribution, and that consumers maximize utility. This generates the logit market shares:

$$s_j(p) = \frac{\exp(\delta_j - \alpha p_j)}{1 + \sum_{k \in \mathcal{J}} \exp(\delta_k - \alpha p_k)} \quad \forall j \in \mathcal{J}, \tag{1}$$

for a vector of prices, p. The share of the outside good is $s_0(p) = 1/(1 + \sum_{j \in \mathcal{J}} \exp(\delta_j - \alpha p_j))$. Throughout, we normalize the market size to unity, which allows us to treat market shares as synonymous with quantity demanded. In dollar terms, consumer surplus is given by

$$CS = \frac{1}{\alpha} \ln \left(1 + \sum_{j \in \mathcal{J}} \exp(\delta_j - \alpha p_j) \right). \tag{2}$$

For any fixed p, consumers benefit from having additional products in the market due to increased variety.

On the supply-side of the model, let firms be indexed by f and the set of firms active in the market be denoted by $\mathcal{F}$. The products in $\mathcal{J}$ are partitioned into a series of sets, where the set $\mathcal{J}^f$ indicates the products sold by firm f. The profit of firm $f \in \mathcal{F}$ is

$$\pi^f(p) = \sum_{j \in \mathcal{J}^f} (p_j - c_j) s_j(p), \tag{3}$$

where c_j is the marginal cost of product j. We assume that each firm maximizes its profit conditional on the prices of other firms. The first order conditions for profit maximization take the form

$$\sum_{k \in \mathcal{J}^f} (p_k - c_k) \frac{\partial s_k(p)}{\partial p_j} + s_j(p) = 0 \quad \forall j \in \mathcal{J}. \tag{4}$$

A price vector that satisfies these first order conditions defines a Bertrand equilibrium.

2.2 Type-Aggregation Representation

We now reformulate the Bertrand logit stage as an aggregative game, following Nocke and Schutz (2018a). The primitives of the firm-level model are the vector of firm-specific types, $\{T^f\}$ $\forall f \in \mathcal{F}$, and the price parameter, α. The type of firm f is defined as $T^f \equiv \sum_{j \in \mathcal{J}^f} \exp(\delta_j - \alpha c_j)$, which represents the firm's contribution to consumer surplus if prices equal marginal costs.

From these primitives, the Bertrand equilibrium can be characterized as a vector of firm-level market shares, $\{s^f\}$ $\forall f \in \mathcal{F}$, a vector of "$\iota$-markups," $\{\mu^f\}$ $\forall f \in \mathcal{F}$, and a market aggregator, H. We define markups below, and let $s^f = \sum_{j \in \mathcal{J}^f} s_j$. The aggregator is defined as $H \equiv 1 + \sum_{j \in \mathcal{J}} \exp(\delta_j - \alpha p_j)$, which also is the denominator from the market share formula of the product-level model (equation (1)).

To characterize equilibrium, we first derive a relationship between the ι-markups and firm-level market shares. Recall that the product-specific price derivatives for logit demand are

$$\frac{\partial s_j}{\partial p_k} = \begin{cases} -\alpha s_j(1-s_j) & \text{if } k = j \\ \alpha s_j s_k & \text{if } k \neq j. \end{cases} \tag{5}$$

Substituting these demand derivatives into the first order conditions of equation (4) for some product j and rearranging gives

$$\alpha(p_j - c_j) = 1 + \alpha \sum_{k \in \mathcal{J}^f} (p_k - c_k)s_k. \tag{6}$$

The right-hand-side of this equation does not depend on the which product $j \in \mathcal{J}^f$ is referenced. This implies that the left-hand-side is equivalent for all products sold by firm f, meaning each firm imposes a common markup. Define the ι-markup of firm f as $\mu^f \equiv \alpha(p_j - c_j)\ \forall j \in \mathcal{J}^f$. Substituting back into equation (6) obtains the relationship

$$\mu^f = \frac{1}{1-s^f}. \tag{7}$$

Turning to the shares, we have $s^f = (1/H)\sum_{j\in\mathcal{J}^f} \exp(\delta_j - \alpha p_j)$ from equation (1), after substituting in for the definition of the aggregator, H. We add and subtract αc_j inside the exponential and apply the definitions of μ^f and T^f to obtain

$$s^f = \frac{T^f}{H}\exp\left(-\mu_f\right). \tag{8}$$

To close the system, the aggregator satisfies an adding-up constraint,

$$\frac{1}{H} + \sum_{f\in\mathcal{F}} s^f = 1, \tag{9}$$

which applies because market shares sum to one. Equilibrium is defined by the shares, markups, and aggregator that satisfy equations (7)-(9).

Nocke and Schutz (2018a) show that a unique equilibrium exists. Thus, there is a unique mapping from the primitives to firm-level shares and markups, and vice-versa. Without knowledge of marginal costs, there is no mapping to product-level prices and shares, though clearly we have that $p_j = \frac{1}{\alpha}\mu^f + c_j$ for $j \in \mathcal{J}^f$. The following equations characterize profit, consumer surplus, welfare, and dead-weight loss:

$$\pi^f = \frac{1}{\alpha}\left(\mu^f - 1\right), \tag{10}$$

$$CS = \frac{1}{\alpha}\ln(H), \tag{11}$$

$$W = \frac{1}{\alpha}\left(\ln(H) + \sum_{f\in\mathcal{F}}\left(\mu^f - 1\right)\right), \tag{12}$$

$$DWL = \frac{1}{\alpha}\ln\left(1 + \sum_{f\in\mathcal{F}} T^f\right) - \frac{1}{\alpha}\left(\ln(H) + \sum_{f\in\mathcal{F}}\left(\mu^f - 1\right)\right). \tag{13}$$

Equation (10) is obtained by rearranging equation (6). As share is a function of the ratio of a firm's type to the aggregator, T^f/H, so too are markups and profit. Equation (11) is obtained by substituting H into equation (2). Equation (12) is simply the sum of consumer surplus and profit. Finally, the first term on the right-hand-side of equation (13) is welfare under marginal cost pricing and the second term is welfare.

Before turning to the analysis of mergers and entry, we characterize some helpful properties of the ι-markup, profits, and prices. First, each firm's markup, profit, and share increase with its type and decrease with the types of competitors:

Proposition 1. *For every firm $f \in \mathcal{F}$, the ι-markup, μ^f; profit, π^f; and market share, s_j $\forall j \in \mathcal{J}^f$ are all increasing in the ratio T^f/H. Furthermore, these objects are also increasing in own-type, T^f, and decreasing in rivals' types, T^g $\forall g \neq f$.*

Proof. See Appendix. The proof follows Nocke and Schutz (2018a) Proposition 6.

Building on the result, consider a merger that enables a set of merging firms, $\mathcal{C} \subset \mathcal{F}$, to maximize joint profit but does not affect primitives. In the type-aggregation representation of the pricing stage, this simply replaces the individual contributions of the merging firms to the vector of types with an aggregated type: $T^m = \sum_{f\in\mathcal{C}} T^f$. The effect of such a merger is to increase the markup, profit, and prices of all firms:

Proposition 2. *For every firm $f \in \mathcal{F}$, the ι-markup, μ^f, and profit, π^f, increase due to a merger among firms in $\mathcal{C} \subset \mathcal{F}$, but the market aggregator, H, and consumer surplus, CS, decrease.*

Proof. See Appendix. The proof follows Anderson et al. (2018), Section 4.3.

2.3 Merger and Entry

Having characterized competition in the third and final stage, we now turn to the entry and merger phases. Without loss of generality, we label firms sequentially according to $f = 1, \ldots, F$, where $F \geq 3$. We refer to the first $F - 1$ firms as incumbents. Firms 1 and 2 are potential merging partners. The timing of the three-stage game follows:

1. Two incumbent firms 1 and 2 decide whether to merge to form a combined firm m. The effect of the merger is to commit these firms to maximize joint profits when setting prices in stage 3. Assume that this merger does not result in any cost or quality efficiencies, meaning that $T^m = T^1 + T^2$.

2. Firm F observes whether a merger occurs in stage 1 and then decides whether to enter the market. If it enters, it incurs an entry cost $g(T^F) > 0$, the value of which is commonly known. We sometimes refer to firm F as the market outsider, and to $g(T^F)$ as an entry barrier.

3. All firms observe whether a merger and/or entry occur in stages 1 and 2. The active firms, which include all incumbents and, if entry occurred, the entrant, form the set $\mathcal{F}$. These firms in $\mathcal{F}$ then set prices simultaneously, consumers make purchasing decisions, and firms earn variable profit according to the Bertrand logit setup described in the previous sections.

Our solution concept is subgame perfect equilibrium (SPE). Firm F enters if it can earn positive profits in the Bertrand pricing stage, taking into account its entry costs and whether a merger has occured. Firms 1 and 2 merge if doing so increases their combined profit in the pricing stage, taking into account the effect of the merger on the decision of the prospective entrant. It follows that a unique SPE of the three-stage game exists, because the merger and entry decisions are made sequentially.

The key case for antitrust analysis is that of *merger-induced entry,* in which entry occurs only with a merger. Applying both Propositions 1 and 2 verifies that a merger does indeed increase the profitability of entry and, as a corollary, there exists some entry cost $g(T^F)$ such that merger-induced entry occurs. Thus, we state formally the following assumption and apply it in our subsequent analysis:

Assumption 1 (Merger-Induced Entry): *Entry is profitable if and only if a merger* *occurs in the first stage, meaning*

$$\pi^F\left(\frac{T^F}{H^{pre}}\right) \leq g(T^F) \leq \pi^F\left(\frac{T^F}{H^{post}}\right), \tag{14}$$

where H^{pre} is the market aggregator with no merger and no entry, and H^{post} is the market aggregator with a merger and entry.

We examine now the incentive to merge in the first-stage of the game. We assume a merger occurs if and only if it increases joint profit:

$$\pi^m\left(\frac{T^m}{H^{post}}\right) \geq \pi^1\left(\frac{T^1}{H^{pre}}\right) + \pi^2\left(\frac{T^2}{H^{pre}}\right), \tag{15}$$

where, in a slight abuse of notation, we let H^{post} and H^{pre} be the market aggregators with and without a merger, incorporating the best-response of the market outsider. Entry matters for the merger decision only if Assumption 1 holds, because otherwise Proposition 2 implies that a merger is always profitable. Further, as entry affects the profit of incumbents through the aggregator, H, under Assumption 1 there exists a cut-off rule such that a merger occurs only if the entrant has a sufficiently small type:

Lemma 1. *Under Assumption 1, there is a cutoff level, T^*, such that a merger occurs in the first stage if and only if $T^F \leq T^*$.*

Proof. The right hand side of expression (15) does not depend on T^F. Proposition 1 implies that incumbent profits decrease as T^F increases, assuming firm F is active. Because profits are a monotonic function of a rival's type, this implies the existence of the cutoff. □

The cutoff type, T^*, characterizes what we refer to as *best-case entry*—an entrant which leaves the merger profit-neutral and thus maximizes consumer surplus conditional on a merger occurring in the first stage. The existence of a best-case entrant, however, raises the question of whether profitable mergers are compatible with merger-induced entry that leaves consumers unharmed. To answer this question, we make use of the type-aggregation representation of the logit model, which provides a simple (but important) intermediary result:

Lemma 2. *If a merger does not affect consumer surplus, then $H^{post} = H^{pre}$, where H^{post} and H^{pre} incorporate the best-response of the market outsider.*

Proof. By inspection of equation (2). □

Thus, the question of interest can be restated as whether merger-induced entry of type $T^F \leq T^*$ (such that mergers are profitable) is sufficient to preserve the aggregator (such that consumers are unharmed). We now provide the main theoretical result:

Theorem 1. *No SPE exists in which a merger is profitable and consumer surplus is unharmed, relative to a scenario in which mergers are prohibited.*

Proof. See Appendix. If entry costs are low enough or high enough that the market outsider has a dominant strategy in the second stage, then the theorem is a straightforward extension of Proposition 2. In those cases, a merger is profitable and decreases consumer surplus. The focus of the appendix, then, is on merger-induced entry, and we find that if $H^{post} = H^{pre}$ then the merger is unprofitable. This relies on the fact that profits can be written in terms of market shares and, given the inverse relationship between shares and the aggregator in equation (8), the merging firm's share cannot simultaneously be high enough to guarantee a profitable merger and low enough to ensure the aggregator does not fall. □

Thus, no two incumbents would pursue a merger in the first stage if they expected entry in the second stage to remedy all harms to consumers in the final stage. Additionally, given that a profitable merger lowers the market aggregator, and no firms experience a decrease in types, Proposition 1 implies that all incumbents' markups are higher in the presence of a profitable merger, even with entry. As marginal costs are held fixed, incumbents' prices must also increase with merger.

Finally, we have focused the analysis on a single market outsider in order to simplify notation and exposition. Theorem 1, however, extends more broadly. Consider an augmented model in which there are multiple outsiders who can enter the market in the second stage, subject to some entry cost. Also assume the incumbents have the option to "reposition" by increasing their type, subject to an investment cost. If a merger is to leave consumer surplus unchanged, then it must be that $H^{post} = H^{pre}$. However, we have established that if $H^{post} = H^{pre}$, then the merger is unprofitable, regardless of the type of entry or repositioning by other firms. Thus, as a corollary, the main result covers multiple entrants and incumbent repositioning.

3 Numerical Analysis

3.1 Model calibration and merger simulation

The primitives of the firm-level model are the vector of types, $\{T^f\}\ \forall f \in \mathcal{F}$, and the price parameter, α. We obtain numerical results using a partial calibration of the model in which only the types are recovered from data on market shares, $\{s^f\}\ \forall f \in \mathcal{F}$. In the calibration routine, we first obtain the market aggregator, H, from equation (9), and the ι-markups, $\{\mu^f\}\ \forall f \in \mathcal{F}$, from equation (7). We then recover firm types from a rearranged equation (8):

$$T^f = \frac{s^f H}{\exp\left(-\mu^f\right)}.$$

With this information, profit, consumer surplus, and welfare are identified up to a multiplicative constant (see equations (10)-(13)).[7]

These outcomes can be compared to alternative scenarios in which (i) two incumbent firms merge and (ii) some new firm enters the market. Each of these alternatives is characterized by a new vector of firm types and, given these types, new equilibrium outcomes can be obtained. The main step in simulation involves using a nonlinear equation solver to recover the shares and the market aggregator. There are $F+1$ nonlinear equations that must be solved simultaneously. One of these is the adding-up constraint of equation (9), and the others are obtained by plugging equation (7) into equation (8), which yields

$$s^f = \frac{T^f}{H} \exp\left(-\frac{1}{1-s^f}\right).$$

With shares in hand, markups, profit, consumer surplus, and welfare are identified up to a multiplicative constant, using the same steps enumerated above for calibration.

3.2 Data Generating Process

In the data generating process, we assume that the merging firms have the same premerger market share; we use shares of 0.01, 0.02, ..., 0.40. We assume the outside good has a share of 0.20, and assign the remaining share evenly to non-merging incumbents.

[7] A full calibration of the model can be accomplished with data on shares and one markup. This allows the price parameter, α, to be recovered from the definition $\mu^f \equiv \alpha(p_j - c_j)$. See also the Nocke and Schutz (2018a) appendix. Then markups, profit, consumer surplus, and welfare also are obtained (not just up to a multiplicative constant). However, these objects are not necessary for present purposes.

Thus, for example, when we allow for five incumbents, one pre-merger equilibrium we examine features shares of 0.10 for each merging firm and shares of 0.20 for non-merging firms. There are 40 share vectors for a given number of incumbents; we consider cases with three, five, six, seven, and nine incumbents.

Given the pre-merger shares, we calibrate the model and simulate the post-merger equilibrium under the alternative assumptions that (i) no entry occurs or (ii) a firm enters the market with a type that makes the merger profit-neutral. The latter case is one of best-case entry.[8] We recover the percentage changes in markups, profit, consumer surplus, and welfare for each of these scenarios; these are identified because the levels are known up to a multiplicative constant.

The main results address the extent to which best-case entry mitigates the adverse consequences of the merger for consumers. To that end, we calculate the percentage of consumer surplus loss that is mitigated by entry:

$$1-\frac{\Delta CS^{\text{entry}}}{\Delta CS^{\text{no entry}}}=1-\frac{\alpha\Delta CS^{\text{entry}}}{\alpha\Delta CS^{\text{no entry}}}, \tag{16}$$

where the left-hand-side of the equation is the object of interest and both the numerator and denominator on the right-hand-side can be recovered with simulation. We also calculate the percentage of the merging firms' price increase that is mitigated by entry:

$$1-\frac{\Delta p^{\text{entry}}}{\Delta p^{\text{no entry}}}=1-\frac{\Delta \mu^{\text{entry}}}{\Delta \mu^{\text{no entry}}}, \tag{17}$$

where again the left-hand-side is the object of interest and both the numerator and denominator on the right-hand-side can be recovered with simulation.

3.3 Numerical results

Figure 1 shows the effects of a merger without entry for a market with six incumbents, for the purposes of building familiarity with the economic environment. The solid, dashed, and dotted lines, respectively, provide the merger effects on markups, profit, and consumer surplus. The effects depend on the pre-merger shares of the merging firms because, within the context of logit demand, larger shares for the merging firms imply greater diversion, more upward pricing pressure, and, ultimately, larger equilibrium price increases. For the smallest merging firms considered, the effects are barely

[8]We search numerically for the type of the best case entrant by simulating the merger with candidate entrants of increasing type until the merger becomes profit-neutral.

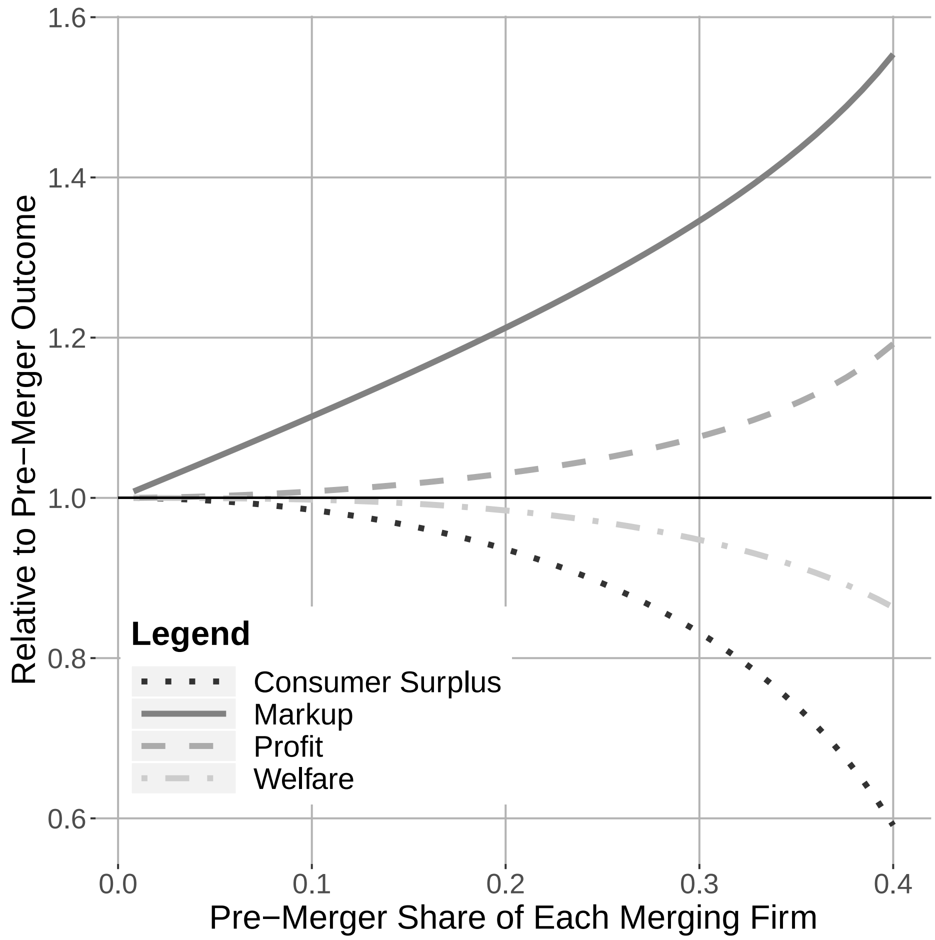

Figure 1: Merger Effects Without Entry

Notes: The figure shows the effects of mergers on consumer surplus, welfare, and the merging firms' markups and profit. These effects are reported as indices relative to pre-merger equilibrium, and depend on the pre-merger shares of the merging firms. There are six incumbents.

discernible, whereas for the largest, markups increase by more than 50 percent, profit increases by 20 percent, and consumer surplus falls by 40 percent.[9]

Figure 2 plots the proportion of the merging firms' price increase that is mitigated by the best-case entrant. The four panels correspond to markets with three, five, seven, and nine incumbents, respectively. In each, the amount of mitigation exhibits a similar pattern: entry does little to offset the price effects of smaller mergers, but becomes more important for larger mergers. For example, with the largest mergers, entry offsets almost 30 percent of the merger price increases. In none of the mergers we examine,

[9]We show results for the case of six incumbents but this does not matter much because prices exhibit mild strategic complementarity with logit demand (e.g., Werden and Froeb (1994); Miller et al. (2016)). Results developed with more or fewer incumbents are available upon request.

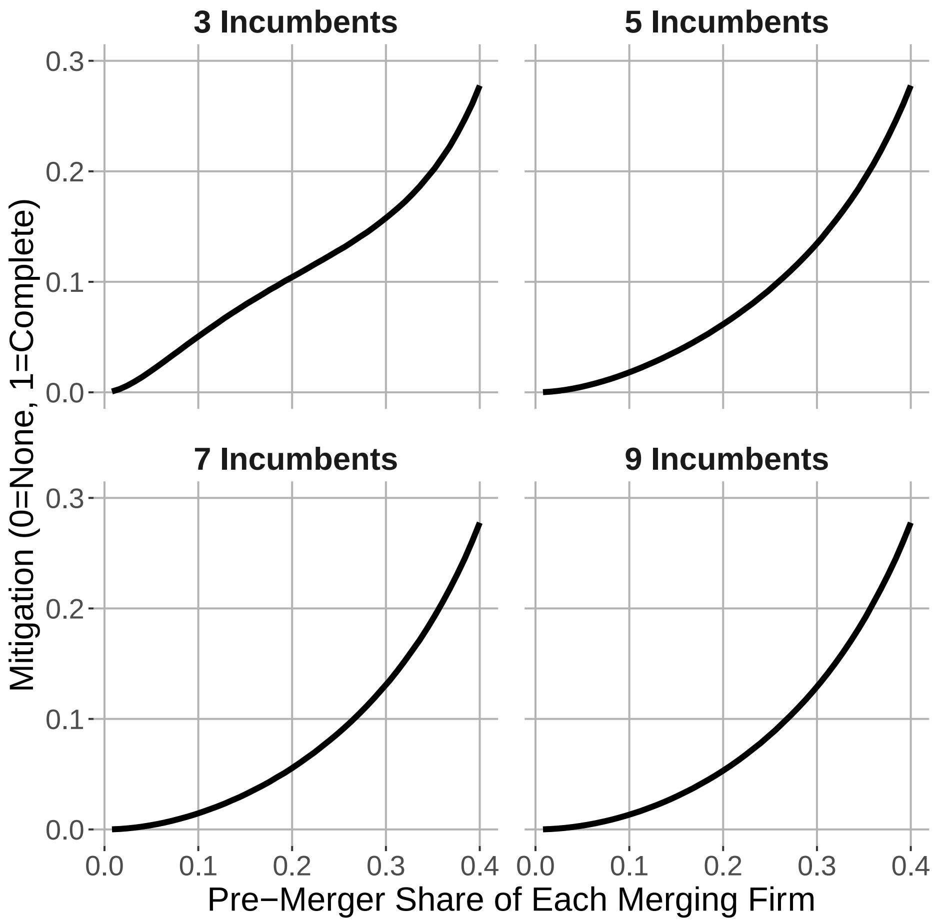

Figure 2: Best-Case Mitigation of Price Increases

Notes: The figure shows the proportion of merging firms' price increases that are mitigated by an entrant that makes the merger profit-neutral. The four panels correspond to models with three, five, seven, and nine incumbents, respectively. Mitigation depends on the pre-merger shares of the merging firms.

however, does entry come close to fully offsetting the price increases.

Figure 3 provides the corresponding analysis of consumer surplus. The lines take a ∪-shape in each panel, such that best-case entry mitigates the least amount of consumer surplus loss for moderately-sized mergers. Further, comparing across panels, mitigation is greater if there are fewer incumbents, and this is most pronounced for small mergers. Finally, the mitigation of consumer surplus loss approaches 50 percent from below in each panel, as the pre-merger shares of the merging firms reach their maximum.

These patterns arise because the entrant contributes to consumer surplus both by improving product diversity and by lowering equilibrium prices. For small mergers, the diversity effect is much more important, as our earlier analysis shows negligible mitigation of merger price increases. Further, from a consumer standpoint there are de-

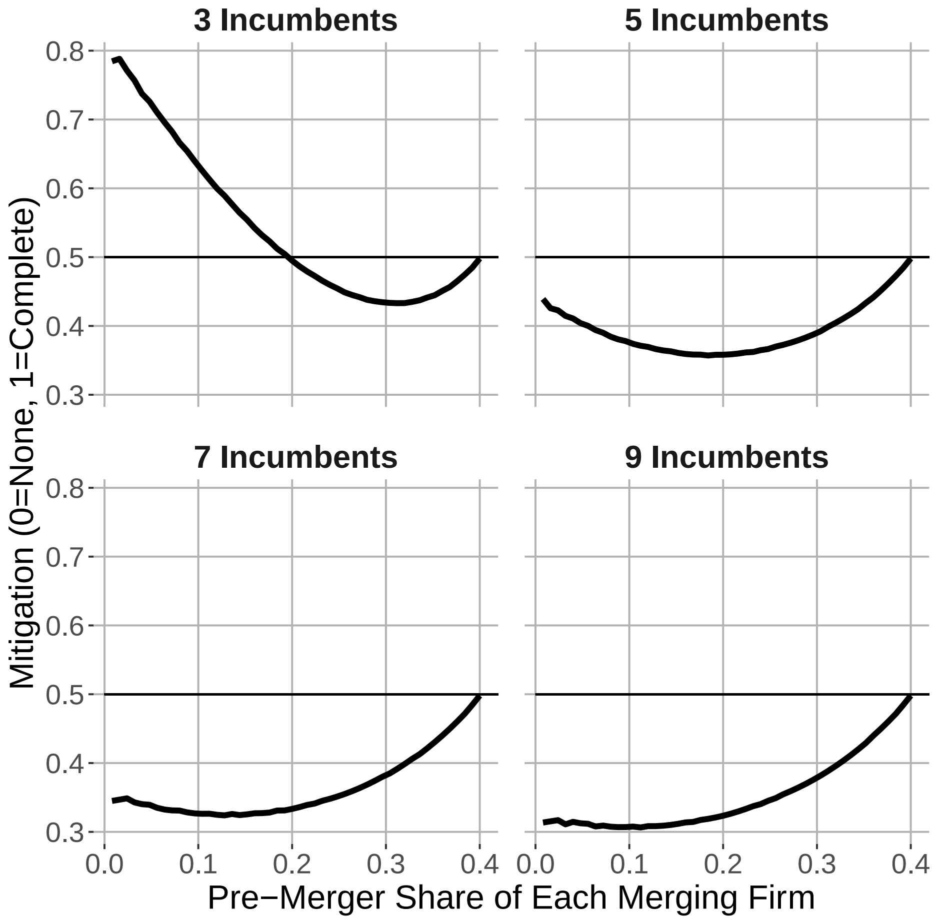

Figure 3: Best-Case Mitigation of Consumer Surplus Loss

Notes: The figure shows the proportion of consumer surplus loss that is mitigated by an entrant that makes the merger profit-neutral. The four panels correspond to models with three, five, seven, and nine incumbents, respectively. Mitigation depends on the pre-merger shares of the merging firms.

creasing returns from diversity, so the number of incumbents matters greatly with small mergers.[10] Moderately-sized mergers produce larger price effects that are (mostly) unmitigated, and the value consumers gain from diversity does not rise commensurately, meaning the overall mitigation of consumer surplus loss is less. For the largest mergers, entrants substantially lessen price effects, leading to greater mitigation of consumer surplus loss than with moderately-sized mergers.

Figure 4 plots the welfare effects of merger. Without entry, welfare decreases due to merger by an amount that increases in the size of the merging firms (see also Figure 1). With best-case entry, mergers between small firms do not reduce welfare, and

[10]From equation (2), with J symmetric firms of type T and marginal cost pricing, we have $CS = (1/\alpha)\ln(1 + JT)$, so consumer surplus grows with the number of firms at a decreasing rate.

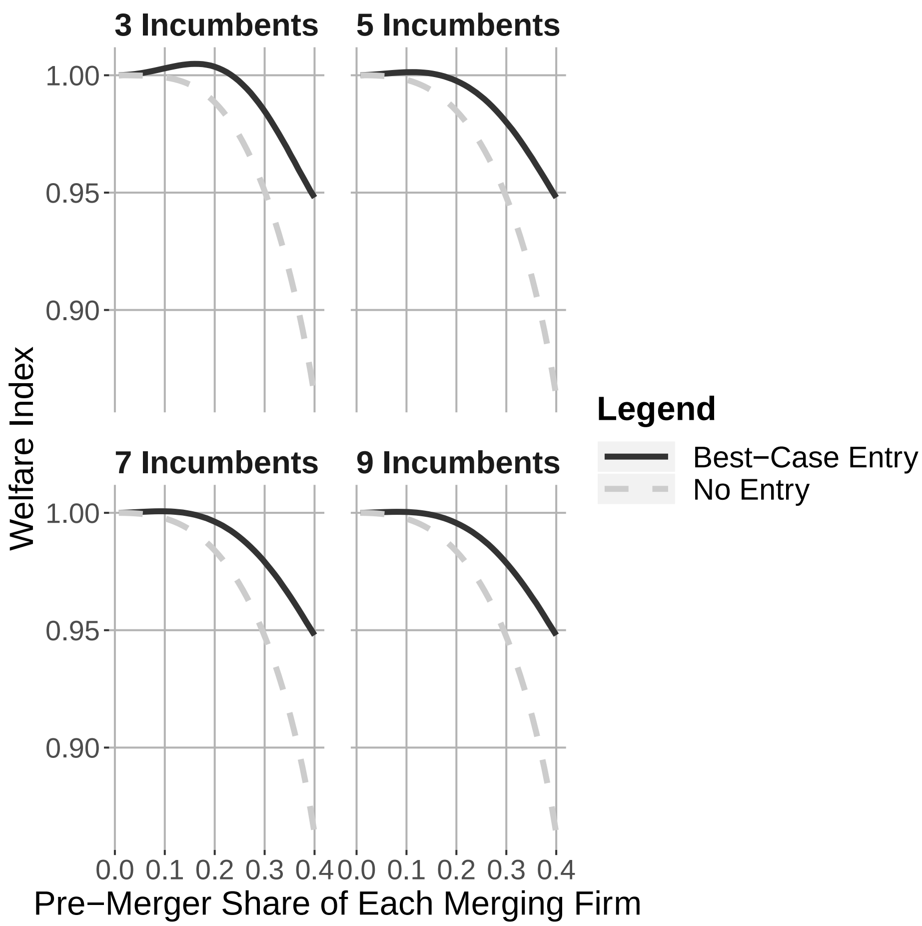

Figure 4: The Welfare Effects of Merger

Notes: The figure shows post-merger welfare with best-case entry and no entry. Welfare is reported as an index relative to the pre-merger value. Results depend on the pre-merger shares of the merging firms.

may actually increase it slightly due the entrant's contribution to product diversity. These increases are most evident if there are fewer incumbents because there improving product diversity has a greater impact on consumer surplus. For mergers between larger firms, best-case entry mitigates about half of welfare losses.

Figure 5 provides some information on the best case entrant. The dashed line plots the ratio of the entrant type to the type of a merging firm. The entrant type is less than 20 percent of the merging firm type for all mergers considered. The solid line plots the entrant share in the post-merger equilibrium, which is is substantially less than half the pre-merger share of a merging firm for all the mergers considered. We interpret these results as suggesting that modest levels of entry can eliminate the profit that otherwise would be obtained from merger. However, such modest entry does little to offset price increases, and may also leave consumer surplus losses mostly unmitigated.

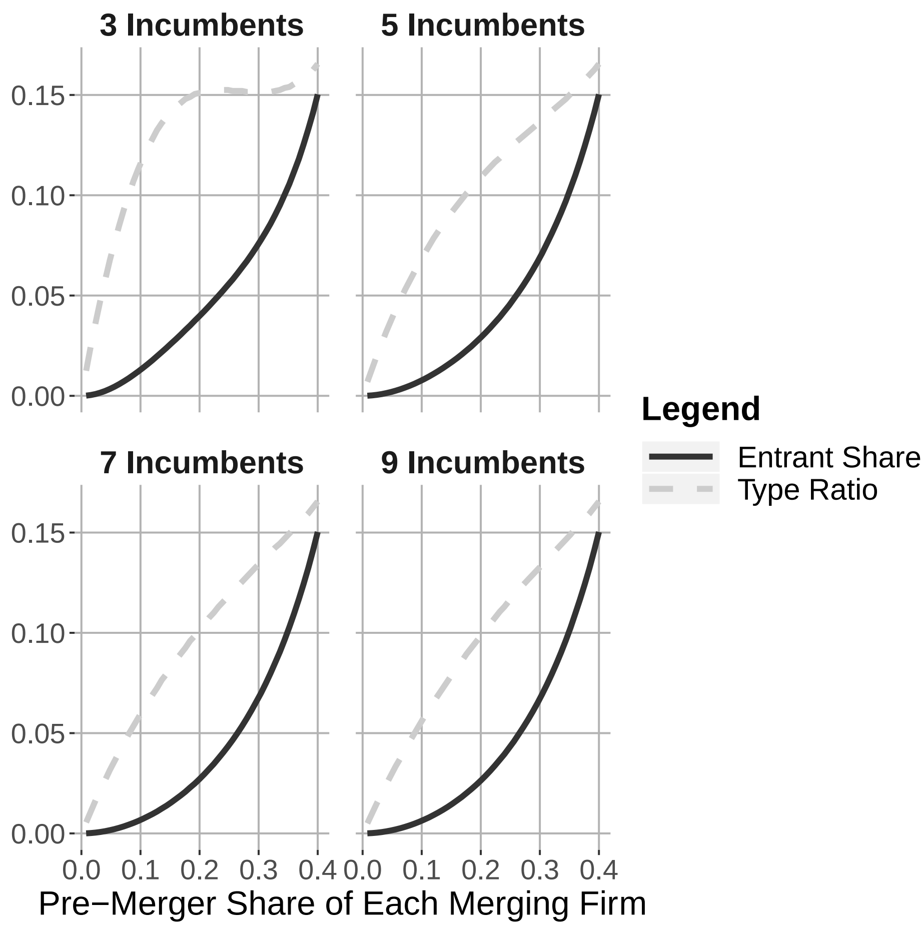

Figure 5: Properties of the Best-Case Entrant

Notes: The figure examines two properties of the entrant that makes a merger revenue-neutral: the entrant share in the post-merger equilibrium and the ratio of the entrant's type to that of a merging firm. The four panels correspond to models with three, five, seven, and nine incumbents, respectively. The properties of the best-case entrant depend on the pre-merger shares of the merging firms.

Finally, the introduction notes that a combination of efficiencies and merger-induced entry can be compatible with profitable mergers that increase consumer surplus. We have searched for parameterizations that meet the following criteria:

(i) Merger with efficiencies alone reduces consumer surplus.

(ii) Merger with efficiencies and entry increases consumer surplus.

(iii) Merger is profitable.

(iv) Entry is more profitable with merger than without.

We have identified many parameterizations that meet these criteria, and even more that do not. One that does features three incumbents with market shares of 0.30, 0.30, and 0.20, corresponding to types of 2.49, 2.49, and 1.46, respectively. Let the first two firms merge with an efficiency that increases the post-merger type from $4.98=2\times 2.49$ to 6.47 (an increase of 30%). Then, with an entrant of type 0.20, the criteria obtain. It follows that entry and efficiencies together can fully mitigate the adverse consumer surplus effects of profitable mergers, even if neither would suffice in isolation.

References

Anderson, S. P., N. Erkal, and D. Piccinin (2018). Aggregative games and oligopoly theory: Short-run and long-run analysis. Working Paper.

Berry, S., J. Levinsohn, and A. Pakes (1995). Automobile prices in market equilibrium. *Econometrica 63*(4), 847–890.

Cabral, L. (2003). Horizontal mergers with free-entry: Why cost efficiencies may be a weak defense and asset sales a poor remedy. *International Journal of Industrial Organization 21*, 607–623.

Deaton, A. and J. Muellbauer (1980). An almost ideal demand system. *The American Economic Review 70*(3), pp. 312–326.

Erkal, N. and D. Piccinin (2010). Welfare-reducing mergers in differentiated oligopolies with free entry. *Economic Record 86*(273), 178–184.

Farrell, J. and C. Shapiro (1990). Horizontal mergers: An equilibrium analysis. *American Economic Review 80*(1), 107–126.

Garrido, F. (2019). Mergers between multi-product firms with endogenous variety: Theory and an application to the RTE-cereal industry. Georgetown University dissertation.

Miller, N. H., M. Remer, C. Ryan, and G. Sheu (2016). Pass-through and the prediction of merger price effects. *Journal of Industrial Economics 64*(4), 683–709.

Miller, N. H. and G. Sheu (2020). Merger simulation and the Horizontal Merger Guidelines. Working paper.

Nevo, A. (2001). Measuring market power in the ready-to-eat cereal industry. *Econometrica 69*(2), pp. 307–342.

Nocke, V. and N. Schutz (2018a). Multiproduct-firm oligopoly: An aggregative games approach. *Econometrica 86*(2), 523–557.

Nocke, V. and N. Schutz (2018b). x An aggregative games approach to merger analysis in multiproduct-firm oligopoly. Working Paper.

Nocke, V. and M. Whinston (2019). Concentration screens for horizontal mergers. Working paper.

Shapiro, C. (2010). The 2010 Horizontal Merger Guidelines: From hedgehog to fox in forty years. *Antitrust Law Journal 77*, 49–107.

Spector, D. (2003). Horizontal mergers, entry, and efficiency defences. *International Journal of Industrial Organization 21*, 1591–1600.

Werden, G. J. (1991). Horizontal mergers: Comment. *American Economic Review 81*(4), 1002–1006.

Werden, G. J. and L. M. Froeb (1994). The effects of mergers in differentiated products industries: Logit demand and merger policy. *Journal of Law, Economics, and Organization 10*(407), 407–426.

Werden, G. J. and L. M. Froeb (1998). The entry-inducing effects of horizontal mergers: An exploratory analysis. *The Journal of Industrial Economics 46*(4), pp. 525–543.

Werden, G. J. and L. M. Froeb (2002). The antitrust logit model for predicting unilateral competitive effects. *Antitrust Law Journal 70*(1), 257–260.

A Appendix for Online Publication

A.1 Proof of Proposition 1

Proof. Let $x^f = T^f/H$. Using equation (7) and substituting in equation (8) defines an implicit function for μ^f in terms of x^f,

$$\mu^f(1 - x^f \exp(-\mu^f)) = 1.$$

After applying the implicit function theorem, we find that

$$\frac{d\mu^f}{dx^f} = \frac{(\mu^f)^2 \exp(-\mu^f)}{1 + (\mu^f)^2 x^f \exp(-\mu^f)}.$$

By inspection, this expression is positive. Furthermore, the derivative of π^f with respect to x^f is equal to $(1/\alpha)$ times the derivative of μ^f, meaning it is also positive if $\alpha > 0$. That is, it is positive so long as consumers earn negative utility from price.

By combining equations (7) and (8), solving for s^f, and taking the derivative with respect to x^f, we find that

$$\frac{ds^f}{dx^f} = \frac{\frac{d\mu^f}{dx^f}}{(\mu^f)^2},$$

which is positive because $d\mu^f/dx^f$ is positive. Additionally, by applying the implicit function theorem to the adding-up constraint (9), we find that

$$\frac{dH}{dT^f} = \frac{\frac{ds^f}{dx^f}}{\frac{1}{H} + \sum_{g \in \mathcal{F}} x^g \frac{ds^g}{dx^g}},$$

which is also positive given that $ds^f/dx^f \; \forall f \in \mathcal{F}$ is positive. Computations for types T^f and T^g give

$$\frac{dx^f}{dT^f} = \frac{1}{H}\left(1 - \frac{T^f}{H}\frac{dH}{dT^f}\right),$$

which can be confirmed as positive after substituting in for dH/dT^f, and

$$\frac{dx^f}{dT^g} = -\frac{T^f}{H^2}\frac{dH}{dT^g},$$

which is negative because dH/dT^g is positive. Applying the chain rule gives the desired results.

$\square$

A.2 Proof of Proposition 2

Proof. By Proposition 1, the markups for the merging firms increase because $T^m > \max_{f \in \mathcal{C}}\{T^f\}$. The corresponding prices must also increase as marginal costs are held fixed. Because the aggregator, H, is decreasing in prices, Proposition 1 also implies that the markups (and thus prices) of non-merging incumbents rise as well, and this further decreases the aggregator. Thus, the value T^f/H is higher for all firms with the merger, so Proposition 1 says that profits increase. Finally, given equation (11), consumer surplus is lower because the aggregator is lower. $\square$

A.3 Proof of Theorem 1

Theorem. *There exists no sub-game perfect equilibrium in which both (i) firms* 1 *and* 2 *merge, and (ii) consumer surplus is unharmed.*

We prove the theorem over a lemma and three propositions. We restrict attention to those sub-game perfect equilibria satisfying (ii). In light of Lemma 2, it suffices to consider the case in which the market aggregator H remains unchanged, that is $H^{\text{pre}} = H^{\text{post}}$. Without loss of generality we normalize this quantity to unity. We start with a refinement of cut-off rule established in Lemma 1:

Lemma A.1. *Under Assumption 1, the merger of firms 1 and 2 increases their joint profits if and only if*

$$s^m \geq \frac{s^1 + s^2 - 2s^1 s^2}{1 - s^1 s^2}.$$

where s^m *denotes the market share of the merged party in the unique differentiated Bertrand equilibrium in the pricing sub-game in which firms 1 and 2 merge, and* s^1 *and* s^2 *are the equilibrium market shares of firms 1 and 2, respectively, in the pricing sub-game in which the firms do not merge.*

Proof. In order for the merger to be jointly profitable, expression (15) must hold. Plugging in for the profit functions using equation (10), we obtain an expression in terms of ι-markups:

$$\mu^m \geq \mu^1 + \mu^2 - 1$$

After substituting in for markups using equation (7), we obtain

$$\frac{1}{1 - s^m} \geq \frac{1}{1 - s^1} + \frac{1}{1 - s^2} - 1.$$

Rearranging this expression gives the desired expression. $\square$

Proposition A.1. *The equilibrium market share function* $S : [0, \infty) \to [0, 1]$ *satisfies the following relation:*

$$T = e^{\ln S(T) - \frac{1}{S(T) - 1}} \tag{1}$$

Proof. By equations (6) and (7) and in light of our normalization, S is implicitly determined via:

$$S(T) = Te^{\frac{-1}{1-S(T)}}.$$

Taking logs and rearranging we obtain:

$$\ln T = \ln S(T) - \frac{1}{S(T)-1},$$

and taking exponentials of both sides yields the desired equation. □

As the merger generates no efficiencies, conditional upon merging, $T^M = T^1 + T^2$. By Proposition A.1, one can express T^M as a closed-form function of equilibrium shares s^1 and s^2 as these are also the ex-ante market shares of the merging parties:

$$T^M = m(s^1, s^2) = \exp\left(s^1 - \frac{1}{s^1 - 1}\right) + \exp\left(s^2 - \frac{1}{s^2 - 1}\right). \tag{A.1}$$

Define the the merger profitability constraint function $c(s^1, s^2)$ via:

$$c(s^1, s^2) = \frac{s^1 + s^2 - 2s^1 s^2}{1 - s^1 s^2}.$$

This allows us to write the condition that the merger be unprofitable (see Lemma A.1) as:

$$S(m(s^1, s^2)) \leq c(s^1, s^2),$$

where now both the left- and right-hand sides are functions of the same variables.

A few observations are now in order. Firstly, define the set of possible (s^1, s^2) pairs as:

$$\Delta = \{x \in \mathbb{R}^2_+ : x^1 + x^2 \leq 1\}.$$

Then $c : \Delta \to [0, 1]$. Moreover, for any $k \in [0, 1]$ we can parameterize any connected component of the k-level set of c as a function of s^1 by suitable restriction of:[11]

$$\tilde{s}(s^1; k) = \frac{k - s^1}{1 - 2s^1 + ks^1}.$$

Of course, as c is symmetric in its variables, the same functional form can be used as a parameterization of the level set as a function of s^2. We note as well that $\tilde{s}(\cdot; k)$ has as

[11] To be explicit, one obtains this formula by starting with

$$\frac{s^1 + s^2 - 2s^1 s^2}{1 - s^1 s^2} = k,$$

and solving for s^2 as a function of K and s^1.

its domain and range $[0, k]$, and satisfies:

$$\tilde{s}(0; k) = k$$

and

$$\tilde{s}(k; k) = 0.$$

Furthermore, by definition it has the property that:

$$c(s^1, \tilde{s}(s^1; k)) = k$$

for all $s^1 \in [0, k]$.[12]

Consider now $m : \Delta \to \mathbb{R}_+$. By definition:

$$S(m(s^1, 0)) = S(T^1) = s^1$$

and

$$S(m(0, s^2)) = S(T^2) = s^2,$$

as $s^i = 0$ implies $T^i = 0$ from the properties of S (namely it is strictly increasing and $S(0) = 0$). Furthermore, because S is strictly increasing, the level-sets of $S(m(s^1, s^2))$ are equal to those of $m(s^1, s^2)$, as $S \circ m$ is just a monotone transformation of m.

Proposition A.2. *For all $k \in [0, 1]$, the function $f_k : [0, k] \to \mathbb{R}_+$ defined via:*

$$f(s^1) = m(s^1, \tilde{s}(s^1; k))$$

is strictly decreasing on $[0, \frac{k}{2-k}]$, strictly increasing on $[\frac{k}{2-k}, k]$, and obtains its maxima jointly at 0 and k.

Proof. Fix $k \in [0, 1]$, and define the path $\gamma : [0, k] \to [0, 1]^2$ via $s^1 \mapsto \big(s^1, \tilde{s}(s^1; k)\big)$. By the chain rule:

$$\frac{dm\big(\gamma(s^1)\big)}{ds^1} = \nabla m(\gamma(s^1)) \cdot \gamma'(s^1),$$

where:

$$\nabla m(\gamma(s^1)) = \left[e^{-\frac{1}{s^1 - 1}} \frac{((s^1)^2 - s^1 + 1)}{(1 - s^1)^2}, e^{-\frac{1}{\tilde{s}(s^1; k) - 1}} \frac{(\tilde{s}(s^1; k)^2 - \tilde{s}(s^1; k) + 1)}{(1 - \tilde{s}(s^1; k))^2} \right]$$

and

$$\gamma'(s^1) = \left[1, -\frac{(k-1)^2}{\big((k-2)s^1 + 1\big)^2} \right].$$

We now verify that f is strictly decreasing on $[0, \frac{k}{2-k}]$. It suffices to show, for all

[12]In other words, the graph of $\tilde{s}(\cdot; k)$ for fixed k is precisely the k-level set of c.

$s^1 \in [0, \frac{k}{2-k}]$:

$$\frac{dm(\gamma(s^1))}{ds^1} \leq 0, \tag{A.2}$$

with equality only at $\frac{k}{2-k}$. Expanding and grouping like terms yields:

$$e^{\frac{s^1 - \tilde{s}(s^1)}{(s^1-1)(\tilde{s}(s^1)-1)}} \leq \left[\frac{(1-s^1)^2}{(s^1)^2 - s^1 + 1}\right]\left[\frac{(k-1)^2}{((k-2)s^1+1)^2}\right]\frac{(\tilde{s}(s^1;k)^2 - \tilde{s}(s^1;k)+1)}{(1-\tilde{s}(s^1;k))^2}. \tag{A.3}$$

Plugging in for $\tilde{s}(s^1, K)$ and simplifying then yields:

$$e^{\frac{s^1 - \tilde{s}(s^1)}{(s^1-1)(\tilde{s}(s^1)-1)}} \leq \left[\frac{(k^2-3k+3)(s^1)^2 - (k^2-2k+3)s^1 + (k^2-k+1)}{((s^1)^2 - s^1+1)\big((k-2)s^1+1\big)^2}\right] \tag{A.4}$$

whose right-hand side is a quotient of bivariate polynomials in (s^1, k).

The left-hand side of (A.4) may be seen by direct computation to be strictly convex in s^1 on $[0, \frac{k}{2-k}]$ for all $0 < k \leq 1$ (and vacuously convex for $k = 0$).[13] As such, for all $k \in [0,1]$, the left-hand side of (A.4) obeys, for all $s^1 \in [0, \frac{k}{2-k}]$:

$$e^{\frac{s^1 - \tilde{s}(s^1)}{(s^1-1)(\tilde{s}(s^1)-1)}} \leq e^{\frac{k}{k-1}} + \frac{2-k}{k}\left(1 - e^{\frac{k}{k-1}}\right)s^1, \tag{A.5}$$

as the right-hand side of (A.5) is simply the linear function (in s^1) connecting the values of the left-hand side at 0 and $\frac{k}{2-k}$. Thus to establish (A.4) it suffices to show:

$$e^{\frac{k}{k-1}} + \frac{2-k}{k}\left(1 - e^{\frac{k}{k-1}}\right)s^1 \leq \left[\frac{(k^2-3k+3)(s^1)^2 - (k^2-2k+3)s^1 + (k^2-k+1)}{((s^1)^2 - s^1+1)\big((k-2)s^1+1\big)^2}\right]. \tag{A.6}$$

Multiplying off the denominator of the right-hand side and grouping terms, (A.6) is equivalent to:

$$0 \leq p(s^1;k) = \sum_{i=0}^{5} a_i(k)(s^1)^i. \tag{$*$}$$

with equality only at $s^1 = \frac{k}{2-k}$, where the polynomial coefficients of p are non-linear

[13]By direct computation:

$$\frac{d^2}{dx^2} e^{\frac{s^1 - \tilde{s}(s^1)}{(s^1-1)(\tilde{s}(s^1)-1)}} = 4\frac{(2-s^1)}{(1-s^1)^4} e^{\psi(s^1;k)}$$

for some $\psi(s^1;k)$, hence its sign depends only on the leading coefficients which are positive and independent of k.

functions of the parameter k given by:

$$\begin{aligned}
a_0(k) &= 1 - \theta(k) - k + k^2 \\
a_1(k) &= -2k\theta(k) + 5\theta(k) - \phi(k) - k^2 + 2k + 3 \\
a_2(k) &= -k^2\theta(k) + 6k\theta(k) - 9\theta(k) - 2k\phi(k) + 5\phi(k) + k^2 + 3 \\
a_3(k) &= k^2\theta(k) - 6k\theta(k) + 8\theta(k) - k^2\phi(k) + 6k\phi(k0 - 9\phi(k) \\
a_4(k) &= -k^2\theta(k) + 4k\theta(k) - 4\theta(k) + k^2\phi(k) - 6k\phi(k) + 8\phi(k) \\
a_5(k) &= -k^2\phi(k) + 4k\phi(k) - 4\phi(k)
\end{aligned} \tag{A.7}$$

and where

$$\theta(k) = e^{\frac{k}{k-1}}$$

and

$$\phi(k) = \frac{2-k}{k}\left(1 - e^{\frac{k}{k-1}}\right)$$

are the intercept and slope respectively of the left-hand side of (A.6).

We will proceed to prove ($*$) as follows. First, we will establish that, for all $k \in [0, 1]$ and all $s^1 \in [0, \frac{k}{2-k}]$ that the third derivative (in s^1) of p, denoted $p^{(3)}$, is negative. This is accomplished by computing the discriminant of $p^{(3)}$ (note $p^{(3)}$ is quadratic in s^1) which is a non-linear function of k. We will show that for all k, the discriminant is negative hence $p^{(3)}$ has no real roots; demonstrating that for all k the leading coefficient of $p^{(3)}$ is negative then establishes the claim. Given that, for all k, $p^{(2)}$ is then everywhere decreasing in s^1, we verify that $p^{(2)}$ is non-negative everywhere by proving it is always positive at the upper bound of the domain. Proceeding analogously, we then show that, for all k, $p^{(1)}$ is non-positive everywhere. Verifying then that $s^1 = \frac{k}{2-k}$ is a root of p establishes ($*$) and proves the result.

1. (**Discriminant of** $p^{(3)} < 0$ **for all** k): We first note that:

$$\begin{aligned}
p^{(3)}(s^1; k) &= 60a_5(k)(s^1)^2 + 24a_4(k)s^1 + 6a_3(k) \\
&= -60\phi(k)(k-2)^2(s^1)^2 + 24\big(\phi(k-2)(k-4) - \theta(k)(k-2)^2\big)s^1 + \cdots \\
&\quad \cdots + 6\big(\theta(k)(k-2)(k-4) - \phi(k)(k-3)^2\big)
\end{aligned} \tag{A.8}$$

which by writing $\phi(k)$ in terms of $\theta(k)$ may be expressed as:

$$p^{(3)}(s^1; k) = \sum_{i=0}^{2} \tilde{a}_i(k)(s^1)^i \tag{A.9}$$

where

$$\begin{aligned}\tilde{a}_0(k) &= 6(k-2)\big[\theta(k)(k-2)k + (1-\theta(k))(k-3)^2\big] \\ \tilde{a}_1(k) &= 24\left[\frac{(2-k)^2}{k}(4-k)\big(1-\theta(k)\big) - (k-2)^2 k\theta(k)\right] \\ \tilde{a}_2(k) &= 60\frac{(k-2)^3}{k}\big(1-\theta(k)\big).\end{aligned} \tag{A.10}$$

The discriminant of $p^{(3)}$ is then:

$$\tilde{a}_1(k)^2 - 4\tilde{a}_2(k)\tilde{a}_0(k) \tag{A.11}$$

which, after simplifying, is strictly negative for all $k \in [0,1]$ if and only if:

$$\big(5\theta(k)-3\big)k^2 + 2\big(\theta(k)-1\big)\big(5\theta(k)-7\big)k - 13\big(1-\theta(k)\big)^2 < 0 \tag{A.12}$$

on $[0,1]$. Let $q(k)$ denote the left-hand side of (A.12). Then:

$$\begin{aligned} q(k) &\leq q(k) + 2k^2 \\ &= \theta(k)^2(10k-13) + \theta(k)(5k^2-24k+26) - (k^2-14k+13) \\ &\leq (1-k)^2(10k-13) + (1-k)(5k^2-24k+26) - (k^2-14k+13) \\ &= 5(k-1)^2(k-3),\end{aligned} \tag{A.13}$$

where only the second inequality requires explanation.[14] In particular, $q(k)$ is strictly less than $5(k-1)^2(k-3)$ for all $k>0$ but may be equal for $k=0$. However, $5(k-1)^2(k-3)$ has only 1 (with multiplicity two) and 3 as its roots. As such, it is strictly negative on $[0,1)$ and equal to zero at $k=1$, and thus (A.12) holds for all $k \in [0,1]$, implying $p^{(3)}$ has no real roots in $[0, \frac{k}{2-k}]$, for any $k \in [0,1]$.

[14]This follows from first noting that if k is 0 or 1 the inequality holds with equality. Moreover, for any given $k \in (0,1)$:

$$\operatorname{sign}\left[(1-k)^2(10k-13) + (1-k)(5k^2-24k+26) - \big(q(k)+2k^2\big)\right]$$

simply equals:

$$\operatorname{sign}\left[\theta(k)(10k-13) - 5k^2 - k + 13\right].$$

By straightforward calculation, $\theta(k)(10k-13)$ is concave on $[0,1]$.This follows from:

$$\frac{d^2}{dk^2}\theta(k)(10k-13) = \theta(k)\frac{(4k-7)}{(k-1)^4},$$

and noting the limit as $k \to 1$ from below is 0. Particularly, it lies above the linear function $f(k) = 13k - 13$ on this domain because f agrees with its values at 0 and 1. But $5k^2+k-13$ is convex as it has a positive leading coefficient; because it coincides with f at $k=0$ and $f(1) > -7$, we have established the second inequality in (A.13).

From this we conclude that $p^{(3)}$ is strictly concave as:

$$a_5(k) = -\phi(k)\big(k^2 - 4k + 4\big), \tag{A.14}$$

which is negative for all $k \in [0,1]$.

2. (**Verifying** $p^{(2)} \geq 0$ **for all** k): From the preceding step, we conclude that, for all $k \in (0,1]$, $p^{(2)}$ is strictly decreasing on $[0, \frac{k}{2-k}]$. To show that $p^{(2)}$ is everywhere non-negative it then suffices to show that for all k it is non-negative when evaluated at $\frac{k}{2-k}$. By direct calculation:

$$p^{(2)}\left(\frac{k}{2-k}, k\right) = \frac{2(k-1)}{k}\Big[\theta(k)(9k^2 - 17k + 10) - (18k^2 - 23k + 10)\Big]. \tag{A.15}$$

Thus to prove the non-negativity of $p^{(2)}$, it suffices to show that the term in the square brackets in (A.15) is non-positive, as the leading $(k-1)$ term is non-positive. To this effect:

$$\begin{aligned} \theta(k)(9k^2 - 17k + 10) &\leq (1-k)(9k^2 - 17k + 10) \\ &\leq (1-k)(9k^2 - 17k + 10) + \Big[k(9k^2 - 8k + 4)\Big] \\ &= (18k^2 - 23k + 10). \end{aligned} \tag{A.16}$$

The first inequality follows from the observation that $\theta(k) \leq (1-k)$ on $[0,1]$.[15] The second inequality follows from the observation that $(9k^2 - 8k + 4)$ is strictly positive everywhere, hence $k(9k^2 - 8k + 4)$ is weakly positive on $[0,1]$. Hence in light of (A.16) we conclude the term in square brackets in (A.15) is non-positive, establishing that $p^{(1)}$ is non-decreasing on $[0, \frac{k}{2-k}]$.

3. (**Verifying** $p^{(1)} \leq 0$ **for all** k): We now prove that $p^{(1)}$ is everywhere non-positive.

[15] For all $y \in (-1, 0]$:

$$\int_1^{y+1} \frac{1}{y+1}\, dt \leq \int_1^{y+1} \frac{1}{t}\, dt,$$

hence

$$\frac{y}{y+1} \leq \ln(1+y).$$

By change of variables $y = -x$, for all $x \in [0,1)$ then:

$$\frac{x}{x-1} \leq \ln(1-x),$$

or

$$e^{\frac{x}{x-1}} \leq 1 - x$$

as claimed. Taking limits verifies the inequality holds with equality for $x = 1$.

In light of the the preceding step showing that $p^{(1)}$ is everywhere non-decreasing, it suffices to show that, for all $k \in [0, 1]$:

$$p^{(1)}\left(\frac{k}{2-k}; k\right) \leq 0. \tag{A.17}$$

By direct calculation again, we obtain that (A.17) is equivalent to:

$$\frac{(k-1)^2}{k(2-k)}\left[\theta(k)(3k^2 - 6k + 4) - (9k^2 - 10k + 4)\right] \leq 0, \tag{A.18}$$

and that again, the sign of the left-hand side is determined by the term in square brackets. But:

$$\begin{aligned} \theta(k)(3k^2 - 6k + 4) &\leq (1-k)(3k^2 - 6k + 4) \\ &\leq (1-k)(3k^2 - 6k + 4) + 3k^3 \\ &= 9k^2 - 10k + 4, \end{aligned} \tag{A.19}$$

where the first inequality follows from the observation $\theta(k) \leq 1 - x$. Thus the term in square brackets in (A.18) is weakly negative, and hence so too is $p^{(1)}$ for any $s^1 \in [0, \frac{k}{2-k}]$.

4. (**Verifying $p \geq 0$ for all k**): In light of Step 3 which showed that, for all $k \in [0, 1]$, $p^{(1)}$ is everywhere non-positive, it suffices to simply observe that:

$$p\left(\frac{k}{2-k}; k\right) = 0$$

to prove $(*)$.

This proves the first claim in (i) of the statement of the theorem. The second claim in part (i), and claim (ii) follow from the symmetry of $\tilde{s}(s^1; k)$ and of $m(\cdot, \cdot)$ in its two arguments.

□

We are now in a position to prove our primary result.

Proposition A.3. *Suppose $H = H^{Pre} = H^{Post}$. Then for all $(T^f)_{f \in \mathcal{F}}$ where firms 1 and 2 are the merging parties,*

$$s^m \leq \frac{s^1 + s^2 - 2s^1 s^2}{1 - s^1 s^2}.$$

Furthermore, the inequality is strict if and only if both s^1 and s^2 are strictly positive.

Proof. Without loss of generality, suppose $s^1 \leq s^2$, and let $K = c(s^1, s^2)$. Then by Proposition A.2, as $0 \leq s^1$:

$$m\big(s^1, \tilde{s}(s^1; K)\big) \leq m\big(0, \tilde{s}(0; K)\big)$$

where this inequality is strict if $0 < s^1$. Hence:

$$\begin{aligned} S\big(m\big(s^1, \tilde{s}(s^1; K)\big)\big) &\leq S\big(m\big(0, \tilde{s}(0; K)\big)\big) \\ &= K \\ &= \frac{s^1 + s^2 - 2s^1 s^2}{1 - s^1 s^2} \end{aligned}$$

as claimed. $\square$